A NEW NUMERICAL APPROACH TO SOLVE TOV EQUATION FOR NEUTRON STARS MATTER

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Abstract Earthly experiments fall short of recreating the incredibly high densities encountered in neutron stars and supernovae. These cosmic events offer an unparalleled opportunity to explore matter squeezed far beyond its usual limits within the nucleus. By piecing together the puzzle using findings from nuclear experiments, astrophysical observations, and powerful theoretical models, scientists can unveil the secrets of this strange, ultra-dense matter. While primarily made of neutrons, neutron stars might contain additional surprise guests – hyperons and quarks. These particles, existing outside the realm of typical atomic nuclei, influence the structure and composition of these stars. The core equations describing these objects, known as the equations of state, are the central theme of this article. We begin with a concise overview of their formation, exploring their defining characteristics and structural peculiarities. Particular emphasis is placed on the crucial role of degenerate neutrons in generating the immense pressure that holds these cosmic giants together. Building upon this foundation, we will delve into the derivation of the Tolman-Oppenheimer-Volkoff (T-O-V) equations. These equations, derived from the principles of general relativity, serve as the cornerstone of our investigation. The T-O-V equations describe the structure of a spherically symmetric, static object composed of isotropic matter under the influence of its gravity. However, solving the T-O-V equations analytically is often intractable. Here, we will explore the application of the power series method as a numerical approach. This method involves expanding the solution as a series of terms with increasing powers of a chosen variable

Keywords: Neutron Stars; Eos of Neutron star; Numerical solution of TOV equation; Power series method of solving TOV equation;

INTRODUCTION

Supernovae give birth to neutron stars. Massive stars reach the end of their lives after roughly 10 million years ($\sim 10^7$years). During this dramatic supernova explosion, the star's core undergoes a violent collapse due to gravity. This immense pressure triggers the formation of a super-dense object at the center, as illustrated in Fig.(1). These compact remnants can take two forms: either a neutron star or a black hole[1]

As the star's core implodes, it initially forms a short-lived "proton-neutron star" with a density exceeding the normal density of atomic nuclei. However, a rapid core bounce occurs, leading to the birth of a true neutron star.

As the name suggests, the interior of a neutron star is dominated by neutrons. At densities around the "nuclear density" ($n \sim n_0$), the proportion of protons to total nucleons (Y_p) is only about 10% ($Y_p = n_p/n$ is 0.1). This means neutrons vastly outnumber protons. The remaining positive charge is balanced by the presence of negatively charged leptons within the star.[1]

While the interior of a neutron star is incredibly hot, reaching temperatures around 10 billion Kelvin ($\sim 10^8$K), this is actually much lower than the energy carried by the individual neutrons (Fermi momenta, $P_F \sim 300$ MeV).

Neutron stars are incredibly compact objects, with masses ranging from 1 to 2 times the mass of our Sun (1 solar mass = M⨀). Their small size, ~10 kilometers in radius, implies an extraordinarily high density. This density is estimated to be around $\sim 7 \times 10^{14}$ g/cm^3 grams per cubic centimeter, which is much larger than any material found on Earth.[3].

The impact of the shock wave's success or failure, generated during the core bounce of a massive star, ultimately determines its fate. If the shockwave effectively pushes back the collapsing core, the star explodes in a supernova and leaves behind a neutron star as a remnant. However, if the shockwave fails, the core continues to collapse, leading to the formation of a black hole.

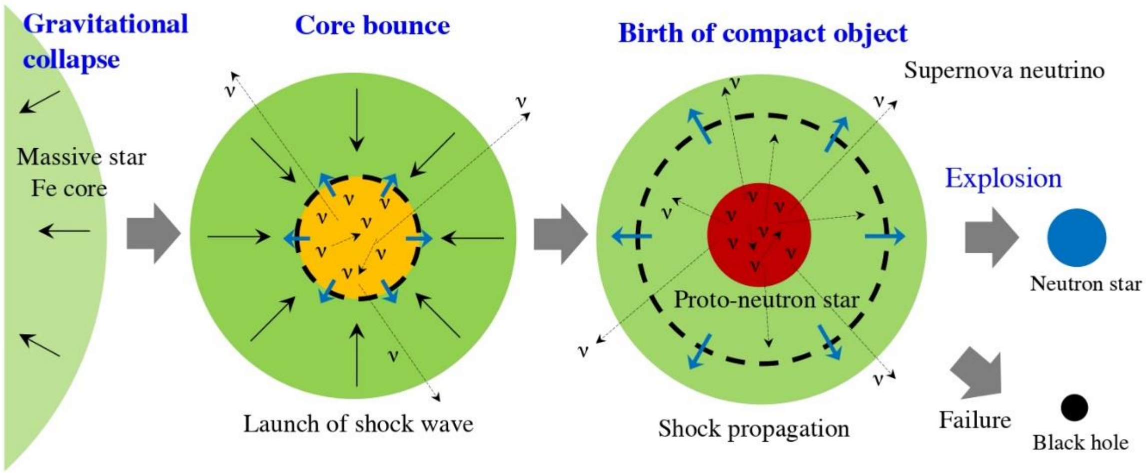

FIG.1:"Schematic diagrams of the evolution of supernova cores from massive stars to compact objects. Initially, the process commences with the gravitational collapse of the Fe core, causing compression of the central core while trapping neutrinos within due to interactions with hot, dense matter. Subsequently, the central core rebounds as a result of the impact of matter around the nuclear saturation density, generating a shock wave. At the core's center, a protoneutron star emerges, characterized by high temperatures and an abundance of neutrinos. The outcome diverges based on whether the shock wave successfully traverses the accreting outer layers of the Fe core: if successful, it triggers a supernova explosion, birthing a neutron star; if unsuccessful, the central object collapses into a black hole"[3]

Pulsars, with their remarkably stable and precise signals that repeat every millisecond or so, provide a strong clue that these objects are incredibly compact. Imagine a large, fast-spinning object - its surface velocity (roughly the product of its spin rate and radius) would exceed the speed of light! This wouldn’t be possible. Luckily, pulsars are often found in binary systems with another star (neutron star, white dwarf, or even a black hole). Using these binary systems and Kepler’s Third Law, scientists can estimate a neutron star’s mass [4-6]. Even more precise measurements involve the slight delay pulsars experience due to general relativity as they travel past their companion’s gravity [7,8]. The heaviest known neutron star, PSR J0740+6620, weighs in at a whopping 2.08 solar masses with a small margin of error [9,10]. This discovery helps us define the minimum stiffness possible for the matter within these stars.

T-O-V EQUATION FOR NEUTRON STAR

Understanding the fundamental properties of an ideal gas composed of electrons and nucleons enables us to model the conditions prevalent in astrophysical phenomena [1, 11, 12]. The neutron stars is a spherically symmetric body of isotropic material in static gravitational equilibrium. Pressure depends on the density of the matter within the star, as described by the equation of state $P(\rho)$. *This equation, known as the Tolman-Oppenheimer-Volkoff (TOV) equation, is like a more general version of Newton's law of gravity, applicable in the extreme environment of a neutron star where Einstein's theory of relativity plays a significant role.*

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}\left(1+\frac{P(r)}{\rho(r)}\right)\left(1+\frac{4\pi r^3 P}{M(r)}\right)\left(1+\frac{2GM}{r}\right)^{-1} \quad (1)$$

The T-O-V Eq.(1) is derived by solving the Einstein equations for a general time-invariant, spherically symmetric metric. To find the mass, $M(r)$, enclosed within a sphere of radius r, we integrate the mass shell at that radius

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \quad (2)$$

We begin with a crucial piece of information: the central density n_c at the heart of the star (represented by $r=0$). An equation of state (EoS) enters the scene, relating pressure P and density ρ within the star. Using the central density n_c, the EoS calculates both pressure and density at various points throughout the star r . We imagine the star composed of thin layers, each called a mass shell. At each point r, a differential equation (2) is employed to determine the contribution of this mass shell to the total mass enclosed within that radius. This equation incorporates the density at that specific point $\rho(r)$ and a factor of $4\pi r^2$ .The integration process of (1) and (2) marches outwards until the pressure reaches zero $P(r = R) = 0$. This point marks the edge of the neutron star, defining its radius R. The total mass of the star is then calculated as the mass enclosed within this radius $M(r = R)$.. By repeating this procedure for various central densities n_c, we gather a collection of data points. These points establish a connection between the mass M and radius R of the neutron star, forming the *M-R* relationship. The specific M-R relationship obtained acts as a unique signature, reflecting the particular EoS used. Theoretically, by observing a neutron star's mass and radius, we can potentially deduce the EoS governing its internal behavior. Using the energy-momentum tensor T^{ab} we can describe a perfect fluid.

$$T^{ab} = (\rho c^2 + P)u^\mu u^\nu + Pg^{\mu\nu} \tag{3}$$

then Einstein Tensor can be written as

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{4}$$

where $g^{\mu\nu}$ are the covariant components of the metric tensor, u^μ is the four-velocity ($u^\mu = \frac{dx^\mu}{d\tau}$), it satisfies the normalization $u^\mu u^\nu = -1$. The star is static so, the three-velocity of the vector field vanishes, staying only the O component, by normalization we can get $u^0 = \frac{1}{\sqrt{-g_{00}}} = 1/a(r)$, where $a(r)$ is a measure of a red shift factor.

$$T^a_b = diag\{-\rho c^2, P, P, P\} \tag{5}$$

After solving Einstein's field equations, we can get the asymptotic flatness condition to yield the three-space metric

$$exp(-2\lambda) = 1 - \frac{2GM(r)}{c^2 r} \tag{6}$$

and total mass inside radius

$$M(r) = 4\pi\left[\int_0^r \rho(r')r'^2 dr\right] \quad (7)$$

The pressure is a source of the gravitational field and the Schwarzschild metric acts as a modification in the denominator of the force law [13]

$$\Phi' = \frac{1}{1 - 2GM(r)/c^2 r}\left(\frac{GM(r)}{c^2 r^2} + \frac{4\pi G}{c^4} rP\right) \quad (8)$$

Solving the Eq.(6) for $\lambda^{`}$ and having the equation for Φ, can take derivative and by multiplying to r can find the $\Phi^{``}$. After obtaining relations for $\Phi^{`}$, $\lambda^{`}$, $\Phi^{``}$, Φ^2 in terms of ρ, P, $P^{`}$, exp2λ, we can express the relations in terms of the included mas $M(r)$. This leads us to derive the equation governing the relativistic hydrostatic equilibrium (1). For EoS $P = P(\rho)$, the T-O-V equations can be integrated from the origin with initial conditions $M(0) = 0$ and $\rho_c = \rho(0)$, until the pressure vanishes in some radius of the Star.

By setting numerical values for pressure and density, we can find the (Eos), then by solving the T-O-V equation, we will obtain other variables. We need to solve a differential equation and find the equation in the form $P = P(\rho)$. To solve it, we will use the power series method [13]. For easy and quick work, use the program "Mathematica" to represent the solution.

NUMERICAL SOLUTION

Our approach is to find a power-series solution to near $r = 0$

$$\frac{dP}{dr} = -\frac{(\rho + P)(m + 4\pi r^3 \rho)}{r(r - 2m)} \quad (9)$$

$$\frac{dm}{dr} = 4\pi r^2 \rho \quad (10)$$

$$m(r) = \sum_j m_j r^j, \quad P(r) = \sum_j P_j r^j, \quad \rho(r) = \sum_j \rho_j r^j. \tag{11}$$

we get zero because $r = 0$. For the EoS near the central density ρ_c would be useful to use expansion $P = P(\rho)$

$$P = P(\rho_c) + \frac{P_c \Gamma_c}{\rho_c}(\rho - \rho_c) + \cdots, \tag{12}$$

To model the impact of interactions and the transition from non-relativistic to relativistic behavior in dense matter, scientists often utilize a specific form called a polytrope.his polytrope takes the form of an equation $P = C\rho^{\Gamma}$, with Γ is a crucial parameter called the adiabatic index. The key point is that the value of Γ is not constant but rather changes at specific densities. These specific densities are referred to as fiducial densitiesx [14].

$d\,(lnP)/d\,(ln\rho)$ evaluated at ρ_c. The second term of the previous equation should be $dP/d\rho$ evaluated at ρ_c . But we can easily see that, $\Gamma_c = \frac{d\,(lnP)}{d\,(ln\rho)}|_{\rho_c} = \frac{\rho_c}{P_c}\frac{dP}{d\rho}$ = To begin with, we can utilize the initial non-negligible terms in each power series and examine the results obtained through a first-order approximation. substituting the power series into Eq.(10) we get

$$\frac{dm}{dr} = m_1 + 2m_2 r + 3m_3 r^2 + 4m_4 r^3 + 5m_5 r^4 + HOT = 4\pi r^2 = (\rho_0 r^2 + \rho_1 r^3 + \rho_2 r^4 + HOT) \tag{13}$$

HOT means Higher Order Terms. By comparing the corresponding powers of r, we can deduce that

$$\begin{array}{ccccc} r^0 & r^1 & r^2 & r^3 & r^4 \\ m_1 = 0 & m_2 = 0 & 3m_3 = 4\pi\rho_0 & 4m_4 = 4\pi\rho_1 & 5m_5 = 4\pi\rho_2 \end{array} \tag{14}$$

we get m_1 and m_2 equal to 0 as well as some expressions for m_3, m_4. Since the second line of the given ordinary differential equation lacks a constant term, there is no component to offset the m_1 term in the first line, leading to the conclusion that $m_1 = 0$. Likewise, since there is no linear term in r in the second line, the equation $2m_2 r = 0$ suggests that $m_2 = 0$ and so on. Consequently, we will disregard m_1 and m_2. We write Eq.(9) as

$$-(r^2 - 2mr)\frac{dp}{dr} = (\rho + p)(m + 4\pi r^2 p) \tag{15}$$

and we'll deal with the LHS and RHS separately [15]. First the LHS gives:

$$\begin{aligned} -(r^2 - 2mr)\frac{dp}{dr} \\ &= -(r^2 - 2m_0 r - 2m_3 r^4 + \cdots)(p_1 + 2p_2 r + 3p_3 r^2 \\ &+ 4p_4 r^3 + \cdots) + HOT \\ &= 2m_0 p_1 r - (p_1 - 4m_0 p_2)r^2 - (2p_2 - 6m_0 p_3)r^3 \\ &- (-2m_3 p_1 + 3p_3 - 8m_0 p_4)r^4 + HOT \end{aligned} \tag{16}$$

The RHS of Eq.(15) gives,

$$\begin{aligned} &(\rho_0 + p_0 + (\rho_1 + p_1)r + (\rho_2 + p_2)r^2 + \cdots)(m_0 + (m_3 + 4\pi p_0)r^3 \\ &\quad + (m_4 + 4\pi p_1)r^4) \end{aligned}$$

$$\begin{aligned} &(\rho_0 + p_0)m_0 + (\rho_1 + p_1)m_0 r + (\rho_2 \\ &\quad + p_2)m_0 r^2[(\rho_0 + p_0)(m_3 + 4\pi p_0) + (\rho_3 + p_3)m_0]r^3 \\ &\quad + [(\rho_0 + p_0)(m_4 + 4\pi p_1) + (\rho_1 + p_1)(m_3 + 4\pi p_0) \\ &\quad + (\rho_4 + p_4)m_0]r^4 + HOT \end{aligned} \tag{17}$$

we can conclude from terms proportional to r_0 that $(\rho_0 + p_0)m_0 = 0$. Given that $\rho_0 = \rho_c$ and, $p_0 = \rho_c$represent the central density and pressure values, respectively (which are non-zero quantities!) we can infer that $m_0 = 0$. Moreover, by examining each power of r individually, we deduce:

r^1	r^2	r^3	r^4	
$(p_1 - \rho_1) = 0$	$p_1 = 0$	$2p_2 = -(\rho_1 + p_0)(m_3 + 4\pi p$	$3p_3 = -m_4(\rho_1 + p_0$	18

By combining the results of Eqs.(18) and (14) and using the EoS in Eq.(12) we find that the first two non-vanishing terms of all three series can be expressed in terms of the central density $\rho_c = \rho_0$. From the EoS we have $p_c = p(\rho_c) = 0$ From Eq.(14]) and from Eq.(18) we have

$$p_2 = -\frac{1}{2}(\rho_c + p_c)(m_3 + 4\pi p_c) \quad (19)$$

$$= -\frac{2}{3}\pi(\rho_c + p_c)(\rho_c + 3p_c) \quad (20)$$

Differentiated ρ with respect to r twice to relate p_2 and ρ_2:

$$2p_2 = \frac{\partial^2 \rho}{\partial r^2}|_{r=0} = \frac{\partial}{\partial r}\left(\frac{\partial\rho\partial\rho}{\partial r\partial r}\right)|_{r=0} \quad (21)$$

because $p_1 = 0$ we see that this equals with

$$\left(\frac{\partial\rho}{\partial p}\right)\left(\frac{\partial^2\rho}{\partial r^2}\right)|_{r=0} \quad (22)$$

and since $\frac{\partial\rho}{\partial p} = c_s^2$ where c_s is the speed of sound we get $\frac{1}{c_s^2 2p_2}$ and using Eq.(12) we have

$$p_2 = \frac{\rho_c}{\Gamma_c p_c} p_2 \quad (23)$$

Finally, to construct the power series for m we substitute these results Eqs. (19), and (23) into the power series for to find

$$m(r) = \frac{4}{3}\pi\rho_c r^3 + \frac{4}{5}\pi\rho_c r^5 + HOT$$
$$= \frac{4}{3}\pi\rho_c r^3 - \frac{8}{15}\pi^2 \frac{\rho_c}{\Gamma_c p_c}(\rho_c + p_c)(\rho_c + 3p_c)r^5 + HOT \tag{24}$$

$$p(r) = p_c - \frac{2}{3}\pi(\rho_c + p_c)(\rho_c + 3p_c)r^2 + HOT \tag{25}$$

$$\rho(r) = \rho_c - \frac{\rho_c}{\Gamma_c p_c}(\rho_c + p_c)\frac{2}{3}\pi(\rho_c + 3p_c)r^2 + HOT \tag{26}$$

This solution will help us predict the Mass and of the Neutron Star. Equation $p(r)$ enables us to find different p_c, ρ_c, Γ_c as initial values, we equate it with 0 $p(r) = 0$ due to finding the distance from the center of the star where pressure is equal to 0, which is Radius *R* of the Star. The next step is to use an equation $m(r)$ to compute the Mass. Below are presented typical values for the radius and mass, derived from the previous parameters within the specified range:

1. $\frac{p_c}{\rho_c}$ in the range $0.4 < \frac{p_c}{\rho_c} < 0.6$
1. p_c in the range $0.001 < p_c < 0.004$
2. Γ_c in the range $2 < \Gamma_c < 4$

and we expect to find solutions for the radius and mass: radius 10-15 km, mass 1,5-2,5 solar masses. The possibility of encountering small errors in the previous values is entirely reasonable and falls within the margin of error. By testing values such as: $\rho_c = 0.000364$, $p_c = 0.000214$, $\Gamma_c = 2.05$ in the expressions for pressure and mass $p(r), m(r)$, we determine that the radius and mass are as follows: *R=13.256* km and *M=2.51216* solar masses.

We can see from Fig.(2) $P(r) - r$ graphics that the pressure drops to 0 when we hit the surface at the radius $r = R$. Even after using typical values from derivation graphics, Fig. (3) – (4) gives the same view. The collated graphical data offers compelling evidence of an inverse relationship between density and radius. A decrease in density results in a proportional increase in radius. Our results in a fine approximation of our experimental values and graphs.

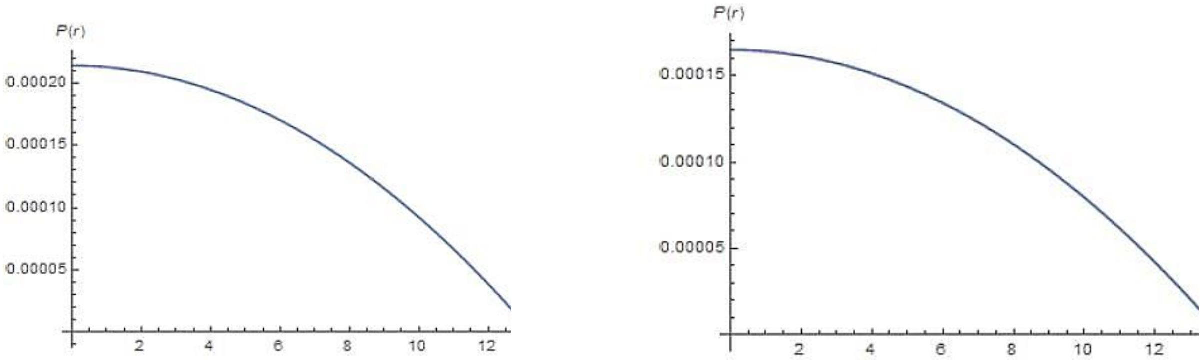

FIG. 2: The P-r graph

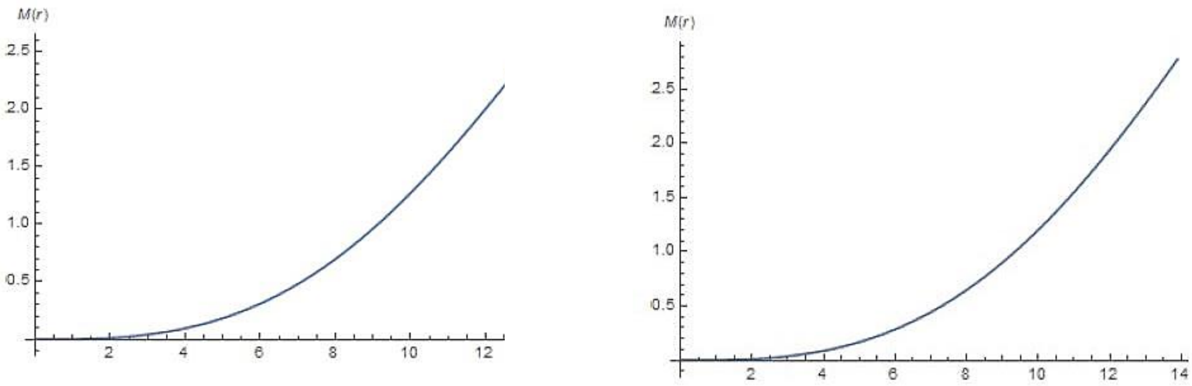

FIG. 3: The M-r graph

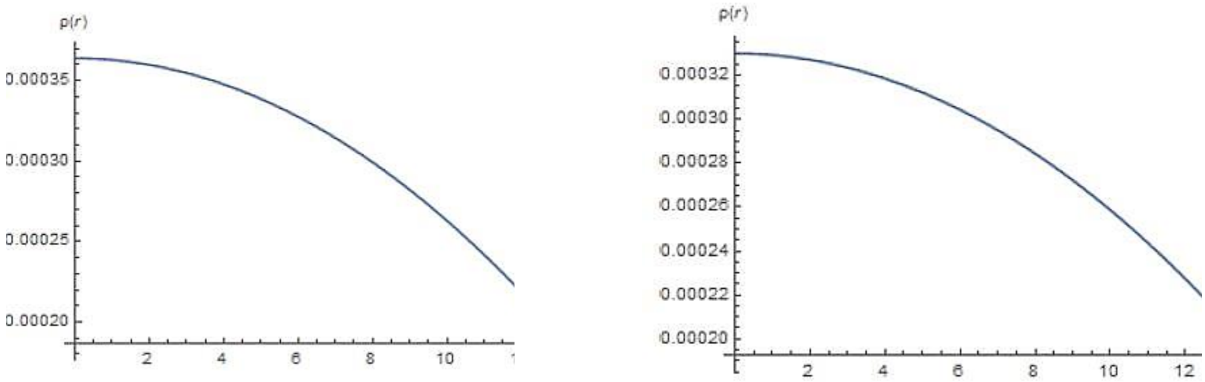

FIG. 4: The $\rho - r$ graph

Conclusion. We have examined and successfully solved the T-O-V equations using the power series method, a reliable approach. It is important to note that we can extract observable data using this solution. The consistency between our derived values and experimental data is excellent, leading us to confidently assert that we have discovered an alternative means to reproduce the attributes of the Neutron Star and its Equation of State.

Determining the radius and mass of the star was achieved by setting the pressure equation to zero, $p(r) = 0$. At this point, $r = R$, yielding the star the radius. Subsequently, inserting this value into the mass equation $m(r)$, we obtain the total mass $M(R)$ in a straightforward and efficient way. It is essential to emphasize the significance of deriving the appropriate expressions for density ρ, basing them on observations of these celestial structures. These expressions serve as initial conditions and vary from one Neutron Star to another. It is worth noting that setting pressure to zero results in a different radius value compared to setting density to zero. Although this discrepancy can be rectified, we opt to utilize $m(r)$ due to its convenience and ease of interpretation.

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