CURVATURE PROPERTIES AND PLANE CURVES

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ANNOTATION: In this article, we will study many properties and characteristics of curves.In addition, we will reveal their important aspects through the tangent vector,the normal vector and the binormal vector.At the end of the article, we will provide the proof of an important theorem related to the theory of curves.Moreover, the essence of the article is revealed by citing specific examples.

KEYWORDS: Regular curves, tractrix, smooth function, plane curvature, tangent vector, normal vector, binormal vector.

INTRODUCTION

We will start with learning the following important aspects.

Parametrized Curves: Assume that $x(t), y(t)$, and $z(t)$ are the coordinate at time t of an "object" moving in space during its action.

The that object's situation at any particular time t is the vector $\alpha(t) = (x(t), y(t), z(t))$.

Let $I \subset R$ denote the interval of time during that we will learn the object's action,so that $\alpha : I \rightarrow R^3$.

It is reasonable to suppose that γ is smooth, that means that every of the three component functions, $x(t), y(t)$, and $z(t)$, separately is smooth in the sense that it can be differentiated any part of times.

To sum up, "a moving object" in space will be shaped through the $n = 3$ situation of a parametrized curve:

But, in this article,we will use from curve parameterized by arc length.

We want to be able to associate to a curve a function that measures how much the curve bends at each point.

Let $\alpha: (a, b) \to R^2$ be a curve parameterized by arc length. Now, in the Euclidean plane any three non-collinear points lie on a unique circle, centered at the orthocenter of the triangle defined by the three points.

For $s \in (a, b)$ choose s_1, s_2, and s_3 near s so that $\alpha(s_1), \alpha(s_2)$, and $\alpha(s_3)$ are noncollinear. This is possible as long as α is not linear near $\alpha(s)$.

Let $C = C(s_1, s_2, s_3)$ be the center of the circle through $\alpha(s_1), \alpha(s_2)$, and (s_3) . The radius of this circle is approximately $|\alpha(s) - C|$. A better function to consider is the square of the radius:

$$\rho(s) = (\alpha(s) - C) \cdot (\alpha(s) - C).$$

Since α is smooth, so is ρ. Now, $\alpha(s_1), \alpha(s_2)$, and $\alpha(s_3)$ lie on the circle so $\rho(s_1) = \rho(s_2) = \rho(s_3)$. By Rolle's Theorem there are points $t_1 \in (s_1, s_2)$ and $t_2 \in (s_2, s_3)$ so that $\rho'(t_1) = \rho'(t_2) = 0$.

Then, using Rolle's Theorem again on these points, there is a point $u \in (t_1, t_2)$ so that $\rho''(u) = 0$. Using Leibnitz' Rule we have $\rho'(s) = 2\alpha'(s) \cdot (\alpha(s) - C)$ and

$$\rho''(s) = 2[\alpha''(s) \cdot (\alpha(s) - C) + \alpha'(s) \cdot \alpha'(s)].$$

Since $\rho''(u) = 0$ we get

$$\alpha''(u) \cdot (\alpha(u) - C) = -\alpha'(s) \cdot \alpha(s) = -1.$$

Now, as s_1, s_2, and s_3 get closer to s, then the center of the circles will converge to a value. Then t_1 and t_2 go to s, so $\rho'(s) = 0$ which forces $\alpha'(s) \cdot \big(\alpha(s) - C_\alpha(s)\big) = 0$. Furthermore,

$$\alpha''(s) \cdot \big(\alpha(s) - C_\alpha(s)\big) = -1.$$

This says that the circle centered at $C_\alpha(s)$ with radius $\alpha(s) - C_\alpha(s)$ shares the point $\alpha(s)$ with the curve α. Furthermore, from the above the tangent to the circle at $\alpha(s)$ is a multiple of $\alpha'(s)$. Thus, this circle, called the osculating circle, is tangent to the curve at $\alpha(s)$. The point $C_\alpha(s)$ is called the center of curvature of α at s, and the curve given by the function $C_\alpha(s)$ is called the curve of centers of curvature.

Definition 1:

The plane curvature of α at s is the reciprocal of the radius of the osculating circle:

$\kappa_{\pm}(s) = \frac{1}{|\alpha(s)-C_{\alpha}(s)|}$.

Theorem1: $\kappa_{\pm}(s) = |\alpha''(s)|$.

Proof: Since $\alpha'(s) \cdot \alpha'(s) = 1$, differentiating gives $\alpha''(s) \cdot \alpha'(s) = 0$. This means that $\alpha''(s)$ is perpendicular to $\alpha'(s)$. Since we have seen that $\alpha(s) - C_{\alpha}(s)$ is also perpendicular to $\alpha'(s)$, there exists a $k \in R$ so that

$\alpha(s) - C_{\alpha}(s) = k\alpha''(s)$.

From above we have

$-1 = \alpha''(s)(\alpha(s) - C_{\alpha}(s)) = \alpha''(s) \cdot k\alpha''(s) = k|\alpha''(s)|^2$.

Thus, $|\alpha(s) - C_{\alpha}(s)| = |k|\,|\alpha''(s)| = \frac{1}{|\alpha''(s)|^2}|\alpha''(s)| = \frac{1}{|\alpha''(s)|}$.

We rarely can symbolically represent a curve as parameterized by arc length. Quite often, a different parameterization is more reasonable. To find the curvature, though, would require that we parameterize by arclength and then differentiate. There is an easier way.

Theorem2:

The plane curvature of a regular plane curve $\sigma(t) = \big(x(t), y(t)\big)$ is given by

$$\kappa_{\pm}(t) = \left|\frac{x''y' - x'y''}{((x')^2+(y')^2)^{\frac{3}{2}}}\right|.$$

About Tractrix line.

Describe the curve followed by a weight being dragged on the end of a fixed straight length and the other end moves along a fixed straight line. The tractrix is the curve characterized by the condition that the length of the segment of the tangent line to the curve from the curve to the y-axis is constant. It has the following equation for a given constant a:

$$x = aln\left(\frac{a + \sqrt{a^2 - y^2}}{y}\right) - \sqrt{a^2 - y^2}.$$

Let the curve begin at $(a, 0)$ on the $x - axis$. Now, we can see that

$$\frac{y'}{x'} = \frac{dy}{dx} = \frac{\sqrt{a^2-x^2}}{x} \qquad (1.1)$$

Square both sides of the equation and simplify

$(x')^2 + (y')^2 = (\frac{a}{x})^2 (x')^2.$

Now, if we differentiate the first equation (1.1), we get

$$\frac{x'y'' - x''y'}{(x')^2} = \frac{-a^2 x'}{x^2\sqrt{z^2 - x^2}}$$

$$x'y'' - x''y' = \frac{-a^2 (x')^3}{x^2\sqrt{a^2 - x^2}}$$

Thus,

$\kappa_{\pm}(x, y) = \left|\frac{-a^2 x'}{x^2\sqrt{z^2-x^2}} \frac{x^3}{a^3 (x')^3}\right| = \left|\frac{x}{a\sqrt{a^2-x^2}}\right|$

Of course, we can integrate Equation 1.1 to get

$$y(x) = \int \frac{x}{a\sqrt{a^2 - x^2}}\, dx$$

A change of variables of the form $x = a\, sin(t)$ gives:

$\sigma(t) = (a\, sin(t), a\, ln(tan(\frac{t}{2}) + a\, cos(t)),$

which gives the plane curvature as $\kappa_{\pm}(t) = \left|\frac{\tan(t)}{a}\right|$.

Also, to parameterize the tractrix by arclength, we need $(x')^2 + (y')^2 = 1,$

Thus, $(\frac{a}{x})^2 (x')^2 = 1$, which gives $x\,' = \pm\frac{x}{a}$.

Let's take $a = 1$ and consider just the case $x\,' = x$. Then, $x(s) = e^s$ from which it follows that

$$\frac{dy}{ds} = \frac{\sqrt{1 - x^2}}{x}\frac{dx}{ds} = \sqrt{1 - (e^s)^2}$$

$$y(x) = \sqrt{1 - (\,e^s)^2} - \, arccosh(e^{-s}\,).$$

This requires that $0 \le e^2 s \le 1$. Take the curve traced out in the opposite direction by replacing s by $-s$. The parameterization is now:

$$\sigma(s) = (\,e^{-s}\,, \sqrt{1 - (\,e^{-s})^2} - \, arccosh(e^{-s}\,)), \qquad s \ge 0.$$

For $a = 1$ we have the plane curvature:

$$\kappa_{\pm}(s) = |\sigma''(s)| = \frac{e^{-s}}{\sqrt{1-(e^{-s})^2}}.$$

Let $\alpha: (a,b) \to R^2$ be a curve. The reverse curve is $\hat{}\alpha: (a,b) \to R^2$ is given by $\alpha\hat{}(t) = \alpha(b - t)$. We wish to distinguish between these two curves.

Definition2

Let e_1, e_2 denote the standard basis vectors in R^2. An ordered pair of vectors $[u,v], u, v \in R^2$ is said to be in standard orientation if the matrix representing the transformation from $[u,v]$ to $[e_1, e_2]$ has a positive determinant.

If $\alpha(s)$ is a regular curve parameterized by arclength, then the unit tangent vector is

$$T(s) = \alpha'(s).$$

Let $N(s)$ denote the unique unit vector perpendicular to $T(s)$ with standard orientation $[T(s), N(s)]$. $N(s)$ is the unit normal vector to α at s. Since $T(s)$ is a unit vector, we see that $T(s) \cdot T'(s) = 0$. Thus,$\alpha''(s) = T'(s)$ must be a multiple of $N(s)$.

Definition3

The directed curvature $\kappa(s)$ of a unit-speed curve α is given by the identity

$$\alpha''(s) = \kappa(s)N(s).$$

Note that since $N(s)$ is a unit vector, we see that $|\kappa(s)| = |\alpha(s)| = \kappa_{\pm}(s)$.

Theorem3: (Fundamental Theorem for Plane Curves)

Given any continuous function $\kappa: (a,b) \to R$, there is a curve $\sigma : (a,b) \to R^2$, which is parameterized by arclength, such that $\kappa(s)$ is the directed curvature of σ at s for all $s \in (a,b)$. Furthermore, any other curve $\bar{\sigma} : (a,b) \to R^2$ satisfying these conditions differs from σ by a rotation followed by a translation.

The proof of this is a very neat, simple proof which uses differential equations.

Proof: From the theorem, we have a function $f : (a,b) \to R^2$ written as $f(s) = (f_1(s), f_2(s))$ satisfying the following system of differential equations:

$(f'_1(s), f'_2(s)) = \kappa(s)(-f_2(s), f_1(s)),$

Note that if f is a solution to this differential equation, then it is a unit-speed curve because

$$\frac{d}{ds}(f_1^{\,2}(s) + f_2^{\,2}(s)) = 2f_1(s)f_1'(s + 2f_2(s)f_2'(s) =$$

$$2\kappa(s)(f_1(s), f_2(s)) \cdot (-f_2(s), f_1(s)) = 0$$

Thus, $|f(s)|$ is a constant and since $|f(c)| = 1, |f(s)| = 1$ for all $s \in (a, b)$.

Lemma1:

If $g(t)$ is a continuous $(n \times n)$ −matrix-valued function on an interval, then there exist solutions,

$F : (a, b) \to R^n$, to the differential equation $F'(t) = g(t)F(t)$.

Applying this lemma, we have a function $g(s)$ given by

$$g(s) = \begin{pmatrix} 0 & -\kappa(s) \\ \kappa(s) & 0 \end{pmatrix}$$

The equation $T'(s) = \kappa(s)N(s)$ becomes $T'(s) = g(s)T(s)$.

Thus, the above lemma gives us the function $T(s)$ for the curve $\sigma(s)$ with the correct curvature. To find the curve $\sigma(s)$ we only need to integrate $T(s)$. We can choose $\sigma(c)$ to be any point in R^2 and we can choose u to be any unit vector in R^2 . Changing u at $\sigma(c)$ involves a rotation. That rotation passes through the differential equation so that another solution would appear as $T^u(s) = \rho_\theta T(s)$, where ρ_θ is a rotation matrix. A translation resets the point $\sigma(c)$ to be any point in R^2. Thus, a second solution $\sigma^u(s)$ must satisfy $\sigma(s) = \rho_\theta \sigma(s) + \omega_0$.

This proves the theorem.

REFERENCES

1. Kreyszig Erwin, Differential Geometry, Leizpig, 1957.

2. L.P. Eisenhart. An Introduction to Differential Geometry, Princeton, 1940.

3. Kristopher Tapp. Differential Geometry of curves and surfaces, Switzerland, 2016.

4. Pavel Tumarkin. Differential geometry lectures, Durham University, 2016.

5. S. Sawin. An Invitation to Differential Geometry, 2015.

6. B–Y. Chen, When does the position vector of a space curve always lie in its rectifying plane?, Amer. Math. Monthly 110 (2003) 147–152.

7. B–Y. Chen and F. Dillen, Rectifying curves as centrodes and extremal curves, Bull. Inst. Math. Academia Sinica 33 (2) (2005)77–90 .

8. F. F. Chen, Introduction to Plasma Physics, Springer, New York 1974.

9. S. Deshmukh, B–Y. Chen, and S. H. Alshammari, On rectifying curves in Euclidean 3–space, Turk. J. Math., 42 (2018) 609–620.

10. R. T. Farouki, Pythagorean-Hodograph Curves: Algebra and Geometry Inseparable, Springer, Berlin 2008.

11. R. T. Farouki, C. Giannelli, and A. Sestini, Helical polynomial curves and double Pythagorean hodographs I. Quaternion and Hopf map representations, J. Symb. Comput. 44 (2) (2009) 161–179.

12. R. T. Farouki, C. Giannelli, and A. Sestini, Helical polynomial curves and double Pythagorean hodographs II. Enumeration of low-degree curves, J. Symb. Comput. 44 (2009) 307–332.

13. R. T. Farouki, C. Y. Han, C. Manni, and A. Sestini, Characterization and construction of helical polynomial space curves, J. Comput. Appl. Math. 162 (2004) 365–392.

14. R. T. Farouki and T. Sakkalis, Rational space curves are not “unit speed,” Comput. Aided Geom. Design 24 (2007) 238–240.